Reverse Engineering of Strong Crypto Signatures Schemes

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Intro

This paper will have the usual classical style of a CryptoReversing Approach, what we going talk about are the ECC also known as Elliptic Curve Cryptography; after a theorial study we will fly to the most common Secured Software Applications with a touch of Hardware Security.ware.

Tools Used

- Ida
- Ollydbg
- A good background of math 'n Cryptography

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Target

- KeygenMe 10 by WiteG

Essay

Why a Reverser should study Cryptography?..many people "erroneously" have the bad "abit" to consider these two disciplines as isolated, but as you will see in the Professional or Recreational Reversing, also the analysis of the most easy and unknown algorithm is done in the same manner, with the same basical assumption and concepts for the inveribility. In Cryptography we will deal with most complex mathematical systems and with more "refined" reversing techniques but..as you will see the Resolution Pattern will be always the same.
Today many professional protections bases their security upon cryptographic algorithms of various kind, and also a big part of the future SSHR (Security-Software/Hardware-Research) will be directed (verso) the realization of complex systems of cryptographical algorithms, maintained by purely mathematical algorithms, and little by little we will need more complex algebraic attacks.

Actually the research is working on new full cryptographic chips, hardened USBs and other hardware that are builded to run specifically a crypto algorithm.

It's necessary to make some other distinction when we talk about hardware cryptography, we have two common systems for the "efficient" computation of an algorithm:

**ASIC Devices:** Application Specific Integrated Circuit, that are specifically builded for maximum risk applications, their power is the efficiency and algorithm can't be changed after the production.

**FPGA devices:** Field Programmable Gate Arrays, which contains arrays of computational elements obtained with a restricted set of instructions. These elements are called logical blocks and are connected with a set of routing resources that could be programmed.

In this last period many security studies on FPGA, revealed us some complex attacks that could be performed, i'm talking about techniques of FPGA exploitation but (s)fortunately this field is not for all :)

Finally...some consideration...there are many yet implemented algos, such as: AES, RSA and ECC. The most interesting in my opinion are the ECC, because they offer a level of security similar to RSA with truly little key sizes, and this is a big quality for hardware devices, ECC are also fast, and are necessary little certificates! (a fundamentall point in mass hardware..PDA, SmartCards, Phones) Introduction to elliptic curve cryptography

The public key cryptosystems most used, are one based on factorization (RSA) and upon discrete logarithm problem (Diffie-Hellman, ElGamal, Schnorr, DES). These algorithms make possible, trusted communications over insecure channels. There are various alternative secure communication systems, one of the is the cryptography based on elliptic curves or more easily ECC, they became a main stream, and this thanks to the numerous advantages that ECC can offer, in easy words extreme flexibility!. We have many proposed elliptic curves for PKC, some of them based on factorization problem, others on DLP. It's important to talk about the fundamental differences between the two "framework", factorization is essentially an academic and except little key sizes there are no big differences with RSA.

More interesting are ECC based on DLP, because the security of these algorithms depends on the redefinition of the classical algorithms used for commo DL problems. This different implementation of classical DLP, drive us to a redefinition of exponentiation, that we can call sub-exponentiation time, if applied to the resolution of elliptical curves. If we look at more general algorithms specifically builded for ECC, we have an exponential time. Sintetically, aspects of the same algorithm assumes different terms, if referred to the efficacy which they have on DLP or ECDLP.

The beautiful story of ECC, begins in the 1984 thanks to Hendrik Lenstra, who coded a factorization algorithm based on the mathematical properties of elliptic curves, called Lenstra Elliptic Curve Factorization. The truly ECC borner in the 1985 by Neal Koblitz and Victor Miller that reimplemted the already known algorithm upon algebraical structures as elliptical curve math and finite fields.

**Basical Math Background - Group Theory**

I've decided to make an approximatively complete discussion of ECC, so in this little chapter i've inserted some elements of Group Theory.

An Abelian Group \((G,\ast)\) is a set \(G\) with assigned binary operation:
that have the following proprieties:

- **Associativity** $a * (b * c) = (a * b) * c$ for all the $a, b, c$ included in $G$
- **Identity Element**, exists an identity element $e$ included in $G$ that $a * e = e * a = a$ for all the $a$ included in $G$
- **Existence of Inverses**, For each $a$ exists an element $b$ called *Inverse* of $a$, so $a * b = b * a = e$
- **Commutativity**, $a * b = b * a$

There are basically two groups, Additive ($+$) and Multiplicative ($*$). This distinction comes from the fact that a group is called additive when its identity element $a$ is 0, while the inverse is -$a$. Multiplicative groups are so called when the identity element is 1, and it's inverse $a^{-1}$. Finally a group is called Finite when $G$ is a finite set, in this case we have also an Order which practically defines the number of elements of $G$.

For example, fixed a prime number $p$, we can easily build a finite group of order $p$, $\mathbb{F}_p = \{0,1,2,...,p-1\}$, obtained from a set of integers. For the precedent observations, we can assume two kind of groups ($\mathbb{F}_p$, $+$) in other words an additive group of modulus $p$ with identity element equal to 0, and also a group ($\mathbb{F}_p^\ast$, $*$) where $\mathbb{F}_p^\ast$ denotes not-null elements from the set that we have considered and as you can see is a Multiplicative group of order $p$-1 and identity element 1.

At this point you may think, exist a group that contains both Additive and Multiplicative operations?..so we have only one mathematical object?..the reply is Yes!! ($\mathbb{F}_p^\ast$, $+$, $*$) called also Finite Fields, now you should know that we can define $G$ as a Multiplicative Finite Group of order $n$ and we can also introduce new elements typical of Finite Fields as the the $g$ element, that the most little positive integer given from $t$ and defined as:

$$g^t = 1$$

called Order of $g$, whose direct and most important consequence is to exist always and to be a divisor of $n$. Another truly important property of groups is the "chain effect", we can indeed define a set as:

$$<g> = \{g^i : 0 \leq i \leq t - 1\}$$

in other words the set of all powers of $g$, which (as you should have understood) is by "it self" a group or better a SubGroup of $G$, and is called Cyclic SubGroup of $G$ generated by $g$. For the nitation of Finite Field, all that we have said it's true also for $G$ writted with Additive Rules, or more precisely the order of $g$ is the most little positive divisor $t$ in $n$, or better:

$$t * q = 0$$

and consequently:

$$<g> = \{i * g : 0 \leq i \leq t - 1\}$$

In the Additive notation $t*g$ assumes the sense of element obtained by adding $t$ copies of $g$, we can also resume with only one definition: If $G$ have a $g$ element of order $n$, then $G$ is a Cyclic Group and $g$ will be called Generator of $G$.

**Finite Fields Arithmetic**

Finite Field Aithmetics is the foundamental basis of each system that uses Elliptical Curves, mainly in cryptography, the correct implementation of all algorithms over Finite Fields, is the dirtst most important step to determine the efficiency and security of an ECC System. There are principally three king of finite
fields:

- **Prime Fields**: formed by prime numbers.
- **Binary Fields**
- **Optimal Extension of the Field**

For each of these fields exists growing implementative difficulties in the sorted order that you can see, it's obvious that is also necessary to implement different algorithms for each kind of Field.

But all algorithms follow one common fundamental concept, the Execution of Arithmetic Operations, INTO and BETWEEN the fields, technically working as Mixer/Connector or as Single Operators.

The Fields, are abstractions or better SubSets called $F$ of the various numerical systems that we know. Into fields are principally possible only two operations, Addition and Multiplication and they have exactly the same properties of the Groups.

In other hands the possible operations with Fields are four, indeed we can add Subtraction and Division, as derived operation types the two principal. Subtraction is defined in additive terms as: $a - b = a + (-b)$, while the Division is defined in terms of Multiplication, as $a/b = a * b^{-1}$ as $b \neq 0$ and $b^{-1}$ inverse element, the element that respect the relation $b * b^{-1} = 1$.

**Finite Prime Fields**: Are written as $\mathbb{F}_p$, where is a prime number $> 3$, defined also as the modulus of $\mathbb{F}_p$, for each integer $a$, ($a \ MOD \ p$) we will have a remainder $r$ between 0 and $p$, the inverse operation, necessary to find $r$ is called **Modular Reduction**.

Let's consider for example the field $\mathbb{F}_{29}$, his elements will be $\mathbb{F}_{29} = \{0, 1, 2, \ldots, 28\}$

So we can define 4 basical arithmetic operations

- $17+28 = 8$ which corresponds to $37 \ MOD \ 29 = 8$ (*Addition*)
- $17-20 = 26$ which corresponds to $-3 \ MOD \ 29 = 26$ (*Subtraction*)
- $17 * 20 = 21$ which corresponds to $340 \ MOD \ 29 = 21$ (*Multiplication*)
- $17^{-1} = 12$ which corresponds to $(17 * 12) \ MOD \ 29 = 1$ (*Inversion*)

With this little example, we can see how basically works arithmetic fields operations and we can also put our attention to an important observation, **the use of finite fields is the principal method to reduce computational complexity in terms of efficiency**, indeed as you should have noticed the simple properties inside an addition ($17 + 28 = 8$) could have enormous "potentialities" if used in cryptography!
As you can see in this is the graph of a generical **Asymmetric** (the case of Elliptic Curve), and you should notice that given a good (long) key-length a computational inversion is too hard, but in the same time, x key-length is easily "workable" by normal computers (forward operation).

**Binary Fields:** Binary Fields are written as $\mathbb{F}_{2^m}$, and called also Finite Fields of Order 2, they could be considered as a vectorial space m in F2 defined by the elements 0 - 1. From basical notions of Linear Algebra exists m elements a, that could be defined with a combination of linear independent vectors that originates the a base (m-1), we will consider this "special" Set as a BitString and over this we will define basical arithmetical operations, the Addition corresponds to the XOR between two BitStrings, Multiplication depends on the choised base. There are many Bases that could be used into $\mathbb{F}_{2^m}$, but the use for computational scopes is reduced because was discovered that some bases are less efficient that other. The choice could be done between Polynomial Basis and Normal Bases, we will works only with the Polynomial Representation. An irriducible polynome f(z) of degree m is choised, irriducible means that f(z) can't be factored as product of polynomes of degree < m. Here follows $\mathbb{F}_{2^m}$ properties:

Let's consider a binary field $\mathbb{F}_2^4$, it's elements are 16 polynomes of max degree 3:

<table>
<thead>
<tr>
<th></th>
<th>$z^2$</th>
<th>$z^3$</th>
<th>$z^3+z^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$z^2+1$</td>
<td>$z^3+1$</td>
<td>$z^3+z^2+1$</td>
</tr>
<tr>
<td>2</td>
<td>$z^2+z$</td>
<td>$z^3+z$</td>
<td>$z^3+z^2+1$</td>
</tr>
<tr>
<td>3</td>
<td>$z^2+1$</td>
<td>$z^3+z$</td>
<td>$z^3+z^2+1$</td>
</tr>
</tbody>
</table>

Possible operations are:

- $(z^3+z^2+1) + (z^2+z+1) = z^3+z$
- $(z^3+z^2+1) - (z^2+z+1) = z^3+z$ (consider that in F2 (-1 = 1))
- $(z^3+z^2+1) \times (z^2+z+1) = z^5 + z + 1$ **FOLLOWS** $(z^5 + z + 1) \mod (z^4+z+1) = z^2+1$
- $(z^3+z^2+1)^{-1} = z^2$ ovvero $(z^3+z^2+1) \times z^2 \mod (z^4+z+1) = 1$

The second member of $\mod (z^4+z+1)$ corresponds to $f(z)=z^4+z+1$

**Generalized Discrete Logarithm Problem**

Let's now study the Discrete Logarithm, we will also see a practical application of Group Theory. In each system based upon the DL we can found some Parameters of Public Domain (p, g and q) were p is a common prime number, q a divisor (also this prime) of p-1, q have a range $[1, p-1]$ and Order q, so we can say that t = q is the most little value that verifies the following relation:

$$g^t = 1 \mod p$$

Now you should see this by other points of view :) , indeed if we make some assumption we can redefine the entire encryption process of DL! suppose indeed that (G,*) is a cyclic multiplicative group of order n that have as generator g, we can "include" the entire algorithm DL into the same G!! . If we consider that Public Domain Parameters are g and n, automatically the private key is an integer x, randomly choised into the range $[1, n-1]$ given by:
\[ y = g^x \]

The problem to determine \( x \), given \( g, n \) and \( y \) is defined as **Discrete Logarithm Problem** (DLP) in \( G \), and a DL system based on \( G \) *should* be untractable but there are some conditions that make it's efficiency attackable. Every two Cyclic Groups of the same order \( n \), these can be considered operatively as the same groups, in better words, we have two idenical boxes with a different content, as immediate effect we can represent the same object in different forms, as computational consequence we will obtain for each representation different efficiency curves (velocity), independently to be from DL or DLP.

What about DL and ECC?..easy we work always with Finite Fields, so researchers rewrited DL into ECC terms :)

**What Elliptic Curves are**

An Elliptic Curve is a plane curve given by: \( y^2 = x^3 + ax + b \)

The principal property (related to cryptographycal scopes) of this kind of curves is that the Set of the points of this curves forms an Abelian Group, which have as Identity element the Infinite. If curve's coordinates are extracted by a Finite Field sufficiently big, the set of their solutions will forms an Abelian Finite Group, you should also remember the DL could be considered as a Set of Finite Cyclic Groups, and the jump to a group of points into an Elliptic Curve is short (if you have clear the previous assumptions) but with a foundamental difference the Increased Complexity, that is the point of force of ECC.

This is is the plot of an Elliptic Curve obtained by \( y^2 = x^3 + ax + b \): 

![Elliptic Curve Plot](image)

**Elliptic Curves in Cryptography**

Basically the Elliptic Curve is only the mathematical architecture because its algebrical properties are used to define the elements of the Set from which is computed the Group. Consider a graph-plot obtained in the plain \( p \times p \), where \( p \) is as usual a prime number, obviously the Field \( \mathbb{F}_p \) that we obtain will go to 0 from \( p-1 \); so algorithmical operations will converge into the points that respects the appartenence condition to \( \mathbb{F}_p \). Elliptical Curves used in cryptography could be of two classes: the First: \( \mathbb{F}_p \) (with \( p > 3 \)) and the Second with \( \mathbb{F}_{2^m} \). In the case of a generical application is used \( \mathbb{F}_q \) (called Extended Optimal Field) where \( q = p \). Into \( \mathbb{F}_p \) elements are essentially integers \( 0 \leq x < p \) derived from Modular Arithmetic operations. The \( \mathbb{F}_{2^m} \) applications are the most complex cause the number of possible representations (the same efficiency concept of DL and DLP) as bitstring of each irreducible polynome \( f(z) \) of degree \( m \).
The couple of affine coordinates \((x,y)\) with \(x, y \in \mathbb{F}_q\) generates an affine plane \(\mathbb{F}_q \times \mathbb{F}_q\), from this specification we can directly obtain the definition of Elliptic Curve \(E\):

An Elliptic Curve \(E\), is the geometrical location of the points of the affine plane, which coordinates satisfies the equation \(4a^3 + 27b^2 \neq 0 \text{(MOD } p\text{)}\); which have as Point at Infinity \(O\), in other words the point where the projective plains encounters the line at infinity. In the most simple case \((p > 3)\) we will have that the \(E(\mathbb{F}_p)\) is the already known \(y^2 = x^3 + ax + b\), where \(a\) and \(b\) (((Appartengono) to \(\mathbb{F}_p\). Let's to a little practical example to clarify :) 

Consider an Elliptic Curve in \(F_7\), you will have as defining equation: \(y^2 = x^3 + ax + b\), so the points will be:

\[E(F_7) = \{\text{Infinity, (0,2), (0,5), (1,0), (2,3), (2,4), (3,3), (3,4), (6,1), (6,6)}\}\]

Now let's consider \(E(\mathbb{F}_{2^m})\) whose defining equation could be written as: \(y^2 + xy = x^3 + ax^2 + b\) where \(a\) and \(b\) comes obviously from \(\mathbb{F}_{2^m}\) and are constants, with \(b \neq 0\) for \(O=(0,0)\) and in other cases \(O=(0,1)\); thanks the Hasse Theorem over elliptic curves, we can quantify the number of points into an elliptic curve by using the following relation: \((\sqrt{q} - 1)^2 \leq |E(\mathbb{F}_q)| \leq (\sqrt{q} + 1)^2\).

**Elliptic Curve Arithmetic**

All cryptographycal mechanism are based over the Elliptic Point Arithmetic, that is the fundamental, practical instrument used to attack/implement ECC. As previously said the points of an Elliptic Curve constitutes an abelian group \((E(\mathbb{F}_q), +)\) that have as usual \(O\) as point defined at the Infinity, this point have the role of Additive Identity, taken two points \(P, Q \in E(\mathbb{F}_q)\) we will have a Third Point defined as \(P + Q\) over \(E(\mathbb{F}_q)\) obtaining as consequence \(P, Q, R \in E(\mathbb{F}_q)\).

Points, as previously said are the elements of an Abelian group, so we can define a set of operations defined in \(G\) that have \(O\) as identity element:

- \(P + Q = Q + P\)
- \((P + Q) + R = P + (Q + R)\)
- \(P + O = O + P = P\)
- \(-P\) such as \(-P + P = P + (-P) = O\)

At the light of this properties, we can introduce two Fundamental Operations over ECs, this is a truly important part..so open your eyes ;)

**Elliptic Curve Addition:** We will use two approaches one Geometrical and one Algebric to better understand what is the real meaning of addition over EC. Exists a rule called of Cord and Tangent that allows us to sum two points of an elliptic curve defined as \(P, Q \in E(\mathbb{F}_q)\) thanks to which we can obtain a Third Point:

\(P + Q = R\)

R is a point of the Elliptic Curve, we can also see (from the Table of Operations) that by defining the negative of \(P = (x, y)\), we will have \(-P = (x, -y)\) for \(P \in E(\mathbb{F}_p)\) and \(-P = (x, x + y)\) for \(E(\mathbb{F}_{2^m})\), so we can define the following rules for the addition:

- if \(Q = O\) then \(P + Q = P\)
- if \(Q = -P\) then \(P + Q = O\)
\[ \text{if } Q \neq P \text{ then } P + Q = R \]

For this last case we can distinguish between:

- **Prime Fields**: \( x_R = l^2 - x_P - x_Q, y_R = l(x_P - x_R) - y_P \) where \( l = \frac{y_Q - y_P}{x_Q - x_P} \)

- **Binary Fields**: \( x_R = l^2 + 1 + x_P + x_Q + a, y_R = l(x_P + x_R) + x_R + y_P \) where \( l = \frac{y_P + y_Q}{x_P + x_Q} \)

You can see algebraically \( \mathcal{E}(\mathbb{F}_q) \) we will obtain a Group with O identity element. in other words an ECC!

Let's consider now the **Geometrical Approach**. Consider \( P \) and \( Q \) as two points of \( E \), trace a line that passes from \( P \) and \( Q \) until intercepts the Elliptic Curve, obviously this will reveal a point that reflected over the x axis will reveal our \( R(\alpha_3, \beta_3) \)

\[
\begin{align*}
P & (2.35, -1.36) \\
Q & (-0.1, 0.836) \\
-\text{R} & (3.89, 5.62) \\
\text{R} & (3.89, 5.62) \\
P + Q & = R = (3.89, -5.62).
\end{align*}
\]

This is only one of the possible Addition cases, now we will see the second case:

**P - P Addition**: \( P + (-P) = O \)

The jointing \( P - P \) is a vertical line that obviously does not "generates" a third point \( R \), so its extension reach the infinity founding the point \( O, P + O = P \)
You will found further references over EC-Addition in the Reverse part

**Duplication of P, 2P:** Duplication can be written as:

\[ 2P = P + P = R \]

From the geometrical point of view, this is equal to build the tangent to the curve into the point P, its prosecution will reveal a point that reflected over x will give us R point:

\[ y^2 = x^3 - 6x + 6 \]

Now come back to some analytical consideration:

- **Prime Finite Field:**
  \[ xR = l^2 - 2xP, \quad yR = l(xP - xR) - yP \]
  \[ l = \frac{3x_P^2 + a}{2y_P} \]
- **Binary Field**: \( x_R = l^2 + l + a, \ y_R = x_P^2 + (l + 1) x_R \) where \( l = \frac{y_P}{x_P} \)

**Duplication of \( P \), if \( y_P = 0 \):** The tangent that passes from \( P \) is ALWAYS the vertical if the component \( y_P = 0 \), consequently if we duplicate a point with this \( y_P \), we will obtain a tangent to the entire Elliptic Curve that obviously will never intercept \( R \) and only intercepting the point \( O \) at infinity:

\[
2P = O
\]

If \( y_P = 0 \), the tangent that passes from \( P \) is always the vertical, and therefore, the point \( P \) is doubled to obtain the point \( O \), the point at infinity.

\[
y^2 = x^3 + 5x - 7
\]

If we want to found \( 3P \) (always in the case of \( y_P = 0 \)) the path is:

\[3P = 2P + P, \ \text{where} \ P + O = P\]

\[4P = O, \ \text{where} \ 2P + 2P = O + O = O\]

\[5P = P, \ \text{where} \ 4P + P = 4P + P = O + P\]

etc..

**Practical Elliptic Curves Operation Samples**

With this last part, you reach a little basical reverser point of view of ECC :) let's see two practical samples of how works EC operations over \( \mathbb{F}_P \):

**Elliptic Curve Addition:** let's consider the point \( P(4,7) \) e \( Q(13,11) \), \( P + Q = (X3,Y3) \) that will be determined as follows:

\[X3 = ( (11 - 7) / (13 - 4) )^2 - 4 - 13 = \]
For y:

\[ Y_3 = 3(4-15) - 7 = -40 = 6 \mod 23 \]

and finally \( R = (15,6) \)

**Elliptic Curve Point Doubling:** let's consider the same point \( P(4,7) \), \( 2P = (X_3,Y_3) \) that will be determined as follows:

\[ X_3 = \left( \frac{3 \times 4^2 + 1}{14} \right)^2 - 8 = 15^2 - 8 = 217 = 10 \mod 23 \]

\[ Y_3 = 15(4 - 10) - 7 = -97 = 18 \mod 23 \]

Our \( R \), will be \( R = (10, 18) \)

Obviously the 23 comes from the field that we choised, \( F_{23} \) ;)

---

**Fundamental Features of EC, and practical ECC generation**

**Order of the Group:** Considering the curve \( E(F_q) \), thanks to the **Hasse Theorem over Elliptic Curves** we can know the number of points of \( E \) included \( O \), here the fundamental relation of Hasse:

\[
|\#E(F_p) - p - 1| < 2\sqrt{p}
\]

Computationally, this is resumed into the **Schoff** algorithm known also as **SEA (Schoof-Elkies-Atkin)**.

The knowledge of the number of points of \( E \) is fundamental for cryptography, in terms of implementability and of security, consider indeed a sum \( P+P+P+... \) sufficiently big, and remember that we're working on Finite Fields, easily we can reach the point \( O \) and this is NOT a good thing, because:

\[ a*P = b*P \text{ for some } a, b \text{ with } b > a \text{ this demostrates the existence of } c*P = O \text{ where } c = b - a !!! \]

**ECC Generation**

This is not nothing more that the Order of the Group, that as you should have understood CAN divide
the order of the group itself! So it's truly important to generate an ECC with a precise order, and we can
do this by using Schoof algorithm (in the previous example obviously this algorithm is used indirectly ;))
or with more complex algorithms, as Complex Multiplication or the more immediate Theorem of Weil,
instead if we're working on Hardware devices we can use the systems of Point Counting as AGM
(Arithmetic Geometric Mean) and SST (Satoh-Skjernaa-Taguchi) that are the most performant ad
spreaded especially over Binary Fields.

Other important conditions for a good ECC is the MOV Condition (Menezes Okamoto e Vanstone),
without this you can fearly reduce the complexity in a logarithmic form! ;)

MOV Condition: Suppose to have an ECC over GF(q) and a point fixed in F, firstly we check that the
first T terms of the sequence \( (q, q^2, q^3, \ldots, q^T) \) are different from \( 1 \mod \text{Prime Order of } F \). T Usually is
choised as

\[ T = \log_2(q)/8 \]

For the entire attack procedure, you can search on CACR website.

Remind also that a field of q elements should have an order different from q or we will have an
Anomalous Elliptic Curve, and that's truly insecure!.

**Elliptic Curves Cryptographic Applications**

And now finally we reached the pratical and most interesting part, what could be the uses of our ECC:

- Use strictly cryptographycal
- Integer Factorizzation
- Primality Proving

The foundamental assumption of EC Cryptography is, *how readapt a certain algorithm based over group
tory with groups based EC Theory*. Here some of the most common ECC Algorithms:

- ECDSA
- ECDH Elliptic Curve Diffie-Hellman
- ECMQV basato su MQV
- ECIES
- EC-KDSA

**Basic ECC Architecture Considerations**

Every implementation of EC applied on crypto algorithms have a basical architecture common to all
algorithms, the foudamental structure is the **Domain Parameters** procedure.

**Domain Parameters**: This structure defines the elliptic curve E in function of the **Field** a **Basic Point G**
and the **Order n** of the field.

Usually is pointed out as: \( D = (q, FR, S, a, b, P, n, h) \)

- \( q \) the order of the Field choised
- \( FR \) (Field Representation) denotes the representation used for the elements of \( \mathbb{F}_q \)
- \( S \) (Seed) in the case the EC needs a PRNG
- \( a, b \) the two coefficients of the defining equation
- \( P \) org G is the **Base Point** or **Generator** (is a cyclic SubGroup), to be cryptographically usable the
  Order of G needs to be the lesser prime number not negative such that \( nG = O \).
- \( n \) is the order n of the Base Point
- \( h \) is the **Cofactor**
Because 'h' is the order of $E(\mathbb{F}_q)$, for the Lagrange Theorem $h = \frac{|E|}{n}$, into cryptographycal operation the cofactor need to be $h \leq 4$ and usually is used $h = 1$.

The Domain Parameters are the most vulnerable elements of an ECC, and are the first parameters that an hypothetical attacker will go to check, it's important from the security point of view that Domain Parameter respects the prescriptions of NIST and SECG; the most classical attacks are the Pohlig-Hellman and Pollard's rho, strictly dependent from the number of points of an elliptic curve.

**Key Pairs:** Are the keys used into ECC, and are strictly related to Domain Parameters, the Public Key is choiced casually by selecting a point $Q$ into the group $<P>$ generated from the point $P$, while the Private Key $d$ is extracted from the following relation:

$$d = \text{LOG}_p (G)$$

**The elliptic curve discrete logarithm problem and ECC Attacks**

The intractability of the DLP is of foundamental importance for the security of every ECC algorithm, we talked briefly about the fact that the problems connected to the DLP are the same for DLP over EC, which is called ECDLP, and can be defined as:

Given a curve $E$ defined over $\mathbb{F}_q$ and a point $P \in E(\mathbb{F}_q)$ of order $n$, and a point $Q$ appartaining to $<P>$, we have to found the integer $l$ included in the range $[1,n-1]$ so that $Q = lP$. This integer $l$ it's indeed the Discrete Logarithm of $Q$ in base $P$ denoted $l = \text{LOG}_p (Q)$.

So you understand how important is the choise of Domain Parameters, necessary to resist against all attacks based on ECDLP. The most known algorithm for ECDLP resolution is the **Exhaustive Search**, that computes the sequence of points $P, 2P, 3P$ until $Q$ is found, the principal disavantage of this algorithm is the low velocity, indeed the Running Time is approximately of $n$, so an $n$ choiced sufficiently big it's a good countermisure against this kind of attacks. Other important attacks are the **Pohlig-Hellman** and **Pollard's rho** that have an Exponential Running Time, precisely of $O(\sqrt{p})$, where $p$ is a prime number sufficiently big, to be protected against this attacks is necessary to choise an $n$ divisible for $p$, such as step-$\sqrt{p}$ will be unusable cause the incrase of computational time.

The basical mechanism of Pollard's Rho it's easy, we have to follows random steps (better defined as **Random Walk**) until $ax, ay, bx, by$ are founded:

$$= ax*B + bx*A == ay*B + by*A$$

$$= ax*l*A + bx*A == ay*l*A + by*A$$

$$= (ax-ay)*l*A == (by-bx)*A$$

$$= (ax-ay)*l == (by-bx) mod NP$$

$$= l == (by-bx)*(ax-ay)^(-1) mod NP$$

(if you're confused with letters just refer to the last crackme analysed in this paper)

There are also other category of attacks, called **Isomorphism Attacks** that try to reduct ECDLP to DLP, most known are the **Weil and Tate Pairing Attacks**, this kind of attacks can be used only in presence of **Anomalous Prime Fields**.
The Elliptic Curve Digital Signature Algorithm

The ECDSA is the correspondent DSA over EC, i choosed ECDSA because is the most diffused also in SW Protections, it's also the most standardized (ANSI X9.62, FIPS 186-2, IEEE 1363-2000 e l'ISO/IEC 15946-2). Let's study the algorithm step by step:

<table>
<thead>
<tr>
<th><strong>ECDSA signature generation</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> D = (q, FR, S, a, b, P, n, h); Private Key d; message m.</td>
</tr>
<tr>
<td><strong>Output:</strong> (r, s)</td>
</tr>
<tr>
<td>1. Select ( k ) appartaining to ( r ) ([1, n-1]).</td>
</tr>
<tr>
<td>2. Computes ( kP = (x_1, y_1) ), nextly converting ( x_1 ) into an integer ( X_1 ).</td>
</tr>
<tr>
<td>3. Computes ( r = X_1 \mod n ). If ( r = 0 ), reselect ( k ).</td>
</tr>
<tr>
<td>4. ( e = H(m) ).</td>
</tr>
<tr>
<td>5. Computes ( s = k^{-1} \cdot (1(e+d*r)) \mod n ). If ( s = 0 ) recomputes ( k ).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>ECDSA signature verification</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> D = (q, FR, S, a, b, P, n, h); Public Key Q; message m, Signature (r, s).</td>
</tr>
<tr>
<td><strong>Output:</strong> Accept / Refuses Signature</td>
</tr>
<tr>
<td>1. Verify that ( r ) and ( s ) are integers included into the range ([1, n-1]), in case the condition is FALSE return Rejected Signature.</td>
</tr>
<tr>
<td>2. ( e = H(m) ).</td>
</tr>
<tr>
<td>3. Computes ( w = s^{-1} \mod n ).</td>
</tr>
<tr>
<td>4. Computes ( u_1 = ew \mod n ) and ( u_2 = rw \mod n ).</td>
</tr>
<tr>
<td>5. ( X = u_1P + u_2Q ).</td>
</tr>
<tr>
<td>6. If ( X = Infinity ) return Rejected Signature.</td>
</tr>
<tr>
<td>7. Converts the coord ( x_1 ) of ( X ) into an integer ( X_1 ).</td>
</tr>
<tr>
<td>8. Computes ( v = x_1 \mod n ).</td>
</tr>
<tr>
<td>9. If ( v = r ) then Signature is Valid, else Rejected Signature.</td>
</tr>
</tbody>
</table>
A Reverse Engineering Approach

Now the Reversing part!, as target i choised crackme 10 of Witeg which implements ECDSA Signature, we will go directly on the ECC part without other techical explanations of the crackme itself.

This crackme implements ECDSA by using MIRACL, it's important to say that truly professional software will never use MIRACL because with few functions you have a fully working ECC Architecture (im'n talking of .NET and CryptoApi).

Let's suppose that we already know that our crackme is based on ECC, so first thing to focus is the Input parameters that the algorithm will receive. We have 4 EditBox, Name and three containing serial (probably the three parameter of every common ECC ;)).

We know also that the second foundamental step of every ECC, is the **Domain Parameter Generation**

```c
miracl *mip;
mip = mirsys(100,10);
mip->IOBASE=16; //The basis is switched to 16, Name and Serial will be in the form 0..9..A...F
secp160r1_a=mirvar(0); //Coeff a
secp160r1_b=mirvar(0); //Coeff b
secp160r1_p=mirvar(0); //Base Point or Generator
secp160r1_n=mirvar(0); //Order of the Base Point
secp160r1_x=mirvar(0); //Coord x
secp160r1_y=mirvar(0); //Coord y
```

As you can see our D will be \( D(a,b,n,x,y) \), there is something different from the canonical D, does not appears the coord \( x \) and \( y \), but don't worry you 'll see in a bit what this mean..now let's study the true core algorithm:

```c
unsigned long lName,lSNX,lSNR,lSNS,snLSB,i,j;
lName = GetDlgItemTextA(hDlg, EDIT_NAME, szName, 0x40); //lName will contain the name
lSNX = GetDlgItemTextA(hDlg, EDIT_X, szSNX, 0x40); //X = first serial inserted
```
lSNR = GetDlgItemTextA(hDlg, EDIT_R, szSNR, 0x40);  //R = second serial inserted
lSNS = GetDlgItemTextA(hDlg, EDIT_S, szSNS, 0x40);  //S = third serial inserted

Ogni curva ellittica necessita di un'inizializzazione:
ecurve_init(secp160r1_a, secp160r1_b, secp160r1_p, MR_PROJECTIVE);
epoint* pointG = epoint_init();  //Initialize a point of ECC called G
epoint* pointH = epoint_init();  //Initialize a point of ECC called H
epoint* pointJ = epoint_init();  //Initialize a point of ECC called J
epoint_set(secp160r1_x, secp160r1_y, 0, pointG);

The first function initializes an Elliptic Curve E of the kind \( y^2 = x^3 + ax + b \mod p \), in other words the classical curve that take the name of Weierstrass's Model, the fourth parameter (MR_PROJECTIVE) specifies if we want to use Affine or Projective coordinates.

hashing(szName, lName, e1);  // e1 = H(Name) review the Signature Verification
lstrcat(szName, szTag);
lName = lstrlen(szName);
hashing(szName, lName, e2);  // e2 = H(Name) this time only the Name that is united to the static Tag

This easy piece of code procudes two messages e1 and e2, using an hashing function H(). The function remain() instead divides a big number with an integer (2 in our case), obtaining so the Integer Reminder of the division \( r/2 \), necessary to give us the bits of the Point Compression.

if ((compare(r, secp160r1_n) < 0) && (compare(s, secp160r1_n) < 0) && 
(epoint_set(x, x, snLSB, pointH) == TRUE))

if (point_at_infinity(pointH) == FALSE)

And now we landend on the first Signature Verification, the checked condition are three, if \( r < n \) and \( s < n \) and if the point H finally appartain to the curve (LSB is referred to the Point Compression resolution) then the execution can continue:

Prima Verifica:

\( \text{xgcd}(s, \text{secp160r1}_n, s, s, s); \)  // \( s = s^{(-1)} \mod \text{secp160r1}_n \) (s can be the w of the point 3 of SignVer proc ;))

\( \text{mad}(e1, s, s, \text{secp160r1}_n, \text{secp160r1}_n, u1); \)  // \( u1 = s*\text{e1 mod secp160r1}_n \)

\( \text{mad}(r, s, \text{secp160r1}_n, \text{secp160r1}_n, u2); \)  // \( u2 = s*r \mod \text{secp160r1}_n \)

ecurve_mult2(u2, pointH, u1, pointG, pointJ);  // \( J = u1*G + u2*H \)
This piece of code should sounds you familiar, indeed is the procedure of Signature Verification that we have seen in the points 3 - 4 - 5, the only difficulty may be in the different letters used, but it's only necessary a bit of attention. The next control is again on the point H, that as you should understood needs to be different from Infinity.

```c
epoint_get(pointJ,x,x);  // x = J.x determine the x component of the point J
if ((compare(x,z)!=0) && (compare(x,r)==0))

it'sfoundamental that xJ is equal between the two messages

Seconda Verifica:

mad(e2,s,s,secp160r1_n,secp160r1_n,u1);  // u1= s*e2 mod secp160r1_n
mad(r,s,s,secp160r1_n,secp160r1_n,u2);  // u2= s*r mod secp160r1_n
ecurve_mult2(u2,pointH,u1,pointG,pointJ);  // J = u1*G + u2*H

The second check is again performed over H, and finally is again founded xJ
if ((compare(x,z)!=0) && (compare(x,r)==0))

If also this check is passed the Signature is Correct!

Finally we are in front of a Signature Verification based on ECDSA, our task is to found a Signature((r,s) and the Private Key (called in this case x) for the two messages e1, e2 that have two hashes of 160 bits.

The Verification are two, one for each message but computationally are equals for both messages, we can indeed see that the kernel of our check is the point J:

\[ J = u1*G + u2*H = \]
\[ = w*e*G + w*r*H = \]
\[ = (w*e + w*r*d)*G \]

Where \( w=s^{-1} \mod n \) and \( H = d*G \)

The J is equally to the already known \( X \) used in the canonical scheme:

\[ X = u1P + u2Q \]

And finnaly we discover that G and H are the famous \( P \) and \( Q \):

As you can see, the true control is indeed to verify the abscisses of the obtained points, that need to be identical for the two messages, may look a complex task, but remind the basical concept of ECC

Everything is a Point!

\( J(x, y) \) and it's projecion \( J(x, -y) \) have indeed the same x, so let's build the Resolving Equation:

\[ w*e1 + w*r*d = -w*e2 - w*r*d \]
\[ = w(e1+e2) = -2*w*r*d \]
\[ d = (n-2)^{-1} * (e1+e2) * r^{-1} \mod n \]
As you can see these are truly easy operations, the only thing is that is necessary a little bruteforce for:

\[ H(r, y_1) = k \cdot G \]
\[ J(x, y_2) = d \cdot G \]

And magically our \( x \) is given by:

\[ x = \text{Lsb}(y) \]

This little "trick" may be truly useful in many many protection schemes.

**ECDLP and Schoof Solving, for Security Patterns**

Now let's study some weakness of Domain Parameters, that make usable the Pollard's Rho / Pollig-Hellman attack to solve DLP.

The resolution of DLP needs necessarily these parameters:

\[ D(P, Q, a, b, p, np) \]

Obviously an ECDLP attack have sense if Domain Parameteres are sufficiently little or badly implemented. In other rare cases DLP can be solved with an easy Bruteforce, as in this case:

```
004013D0 push eax
004013D1 push 0Ch
004013D3 mov [esp+0A0h+var_58], 0
004013D8 mov esi, ecx
004013DA call bytetobig ;Build a Bignumber A, using some letters that cames from a serial
```

```
004013E7 push ecx
004013E8 push ebx
004013E9 push ebp
004013EA push edi
004013EB call powmod ;27AB8CB1F847BBB412CA33^A mod C3CEAB067BEC3F369EB2103
```

```
0040140C push eax
0040140D push ecx
0040140E call _compare ;Powmod == 7DC79E80D9CBBDB7DB291E43C
```

This piece of code is taken from a crackme truly easy but in the same time you can see the hint to use DLP Braking, a BigNumber A is builted using some characters of the Serial and finally with a Powmod
27AB8CB1F847BBBC412CAA33^A mod C3CEAB06781ECF3B69EA2103, it compares the result obtained with 7DC79E80D9CBBDB7DB291643C, consequently we have to found the correct value of A, for this we have to use DLP.

We are in the case of the order of 27AB8CB1F847BBBC412CAA33 over the field \( F(C3CEAB06781ECF3B69EA2103) \) is \( C3CEAB06781ECF3B69EA2102 = 2*3*1D*78B*46FA51*89C040BCD81E05 \) so have sense to use Pollard's Rho and il Polhig-Hellman in combo.

Let's see another case:

```assembly
004013F0 push ebp
004013F1 push esi ;Serial length
004013F2 call inttobig ;Transform the length of the serial into BigNum

[...]
004013FD push ebp
004013FE push edi
004013FF call powmod ;27AB8CB1F847BBBC412CAA33^B mod C3CEAB06781ECF3B69EA2103

[...]
00401426 push edx
00401427 push eax
00401428 call _compare ;Powmod == 4A2BEE4544261D982D959675
```

The algorithm generates a BigNumber B that contains the length of Serial and as usually executes a \( \text{Powmod} \) \( 27AB8CB1F847BBBC412CAA33^B \mod C3CEAB06781ECF3B69EA2103 \) and next step is the comparision with \( 4A2BEE4544261D982D959675 \), what we can do in this case?...Polhig-Hellman..no..it's not necessary B is a BigNumber, but it's dimension is truly little (it expresses only the length of the Serial) and consequently B can be founded with a basical Bruteforce.

After obtaining A and B, two BigNums \( X_1 \) and \( X_2 \) are generated, and next by using the curve \( y^2 = x^3 + x \) into the field \( F(ACC00CF0775153B19E037CE879D332BB) \) and with A e B, are determined:

\[
P = X_1A + X_2B
\]

\[
c = X_2 - P.x
\]

next by checking that \( c \) begins with "TMG-" and finally by jointing \( c \) with Name we will have our serial:

For the resolution \( X_1 \) and \( X_2 \) are arbitrarly choised:

\[
P = X_1A + X_2B
\]

\[
X_2' = c + P.x
\]

It's now necessary to find \( X_1' \) to satisfy the following \( X_1'A + X_2'B = X_1'A + X_2'B \) the equivalence criteria is based on the concept that we have to obtain the same abscis of P, and also that A and B are of the same order, so we have to compute \( l \) so that \( l^A = B \), in other words we have to solve the ECDLP. With Schoof we can also know that the Curve have \( ACC00CF0775153B19E037CE879D332BC \) points
and consequently thanks to Lagrange Theorem we have the order of A and B, as:

\[ 56606783\text{BA8A9D8CF01BE743CE9995E} = 2^3\cdot 7\cdot D\cdot 7'\cdot D'\cdot D'\cdot 3\cdot 1\text{DF75}\cdot 5978\text{F1F374C47}\cdot 5\text{F73FD8D3} \]

..and as a fairy tale the good old Pohlig Hellman will guide us :P, it's hint is to compute the ECDLP as \( F(2), F(3), F(7), F(D^2), F(7F), F(D3), F(1\text{DF75}), F(1\text{F374C47}), F(5\text{F73FD8D3}) \) which is better than computing the entire BigNumber.

fonded \( l = 1212121255555\text{ABCDEFABCDEF9999999} \) follows that:

\[
\begin{align*}
X2*A + X1*B &= X2'A + X1'*B = \\
&= X2*l*B + X1*B = X2'*l*B + X1'*B \\
&= X2*l*B + X1*B = X2'*l*B + X1'*B \\
&= X2*l + X1 = X2'*l + X1' \mod NP
\end{align*}
\]

ed infine..

\( X1' = X2*l + X1 - X2'*l \mod NP \)

This is the end, surely in future I'll expand this paper with other practical examples and theory.

Regards,

Giuseppe 'Evilcry' Bonfa'